An Application of Number Theory to the Splicing of Telephone Cables *

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The consideration of a simple and practical splicing scheme for minimizing the recurrence of same-layer adjacencies among telephone circuits in long cables leads to a problem in Number Theory whose solution calls for some extension of the previous work in this field. The solutions for numbers not greater than 139 have been computed, and a table of these is included.

SOME time ago in connection with the placing of a long telephone cable the writer had occasion to attempt the specification of a splicing scheme designed to minimize the recurrence of same-layer adjacencies among the telephone circuits as they threaded their way through successive lengths of the completed cable. The task, superficially so simple, proved to be one of most intriguing difficulty, and the pursuit of the solution led a confused investigator stumbling into the province of number theory. That speculation upon an art so mundane as that of telephone cable splicing should have led to a proposition in the oldest and most neglected branch of mathematics seemed to be especially worthy of note, for few applications so practical have been found. In the course of the investigation certain small ground apparently was covered for the first time. It was felt, therefore, that the story would be of passing interest alike to the mathematician and to the engineer.

The present standard cables for long distance telephone service are manufactured as a series of concentric layers of conductor units contained within a cylindrical sheath. The conductor units are either pairs of quads of wires. The layers are one unit in thickness, and successive layers either spiral in opposite directions of rotation, or in the same direction but with different pitches. The feature of importance to this discussion is that in an unbroken length of cable any one conductor unit will experience shoulder-to-shoulder adjacency throughout this distance with the two conductor units lying on either side in the same layer, and its experience with these two conductor units will be unique. Cables usually are manufactured in uniform lengths of from 750 to 1000 feet, and a longer cable is made up from a succession of such

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lengths spliced end-to-end. At each splice point a large number of different splices is possible among conductor units. In general, wire-to-wire splices are not made, and considerable mixing up is achieved. For reasons which need not be given here it is considered desirable from the standpoint of crosstalk control that each telephone circuit experience the minimum amount of same-layer adjacency with every other telephone circuit.

For the purposes of this discussion it will suffice at present to consider the cross-section of a cable as a simple closed sequence of N consecutively adjacent units. As an example, the array presented by a circular picket fence would be of this character. Each conductor unit in a cable is identifiable, and it will be assumed that each has been "tagged" with one of the numbers $1, 2, 3, 4, \cdots, N$ in such sequence that units bearing consecutive numbers lie adjacent—remembering that unit No. 1 and unit No. N also lie adjacent. While this simple picture of the cable cross-section is representative truly of only a single layer structure, still the results of a study of it may be fitted to apply to practical cases. Schemes for accomplishing this will suggest themselved to the practical worker, and their discussion here would burden this presentation unduly.

Consider now two consecutive lengths in a completed cable and focus attention upon a conductor unit in one of these. At the splice point this conductor unit may connect to any one of the conductor units in the second length, and the two conductor units which lie alongside the latter in the same layer in the second length may connect to any two of the $N - 1$ remaining conductor units in the first length. As an extended conductor unit traverses the completed cable, then, it may experience same-layer adjacency successively with any possible combinations two at a time of the other extended conductor units, and in any order, sequence, or repetition of these as determined by the splicing scheme that is used. Since there can be but $[(N - 1)/2]$* totally different combinations two at a time of $N - 1$ different objects it is evident that $[(N - 1)/2]$ successive cable lengths is the maximum possible number for an extended conductor unit to traverse without incurring repetition of at least one of the same-layer adjacencies that occurred in the first of these lengths.

Any splicing scheme that is devised for practical use must embody the utmost in simplicity. For this reason it is considered highly desirable (1) that the required results be achieved through repetition of the same splicing instruction at consecutive splice points, and (2)

* The symbol $[x/y]$ means the greatest integer not greater than x/y.

that this instruction follow the simplest possible system—e.g., any two adjacent conductor units in one length of cable shall connect to two conductor units having a constant separation in count in the next length. The exposition which follows makes no attempt to solve the general problem, and seeks only to establish the results which can be realized when the above two simplifying restrictions are imposed. At the conclusion is added a description of a minor and acceptable deviation from the second restriction which will enable the practical worker to supplement these results and achieve the maximum possibilities in a number of cases sufficient for his needs. The problem now will be formulated.

$1 \to 1$	$1 \to 1$	$1 \to 1$
$2 \to 2$	$2 \to 3$	$2 \to 4$
$3 \to 3$	$3 \to 5$	$3 \to 7$
$4 \to 4$	$4 \to 7$	$4 \to 10$
$5 \to 5$	$5 \to 9$	$5 \to 2$
$6 \to 6$	$6 \to 11$	$6 \to 5$
$7 \to 7$	$7 \to 2$	$7 \to 8$
$8 \to 8$	$8 \to 4$	$8 \to 11$
$9 \to 9$	$9 \to 6$	$9 \to 3$
$10 \to 10$	$10 \to 8$	$10 \to 6$
$11 \to 11$	$11 \to 10$	$11 \to 9$
Fig. 1	Fig. 2	Fig. 3

The three tabulations exhibited in Figs. 1, 2, and 3 show possible ways of splicing two pieces of eleven-unit cable together in systematic fashion. The left-hand columns indicate the consecutively adjacent conductor units in the first or reference piece of cable (remembering that No. 1 and No. 11 are adjacent), and the numbers opposite in the right hand columns indicate the conductor units in the second piece of cable to which splice is made. No importance attaches to the splicing of unit No. 1 to unit No. 1 in each instance. This is simply one of eleven possible "starts," and from the point of view of this discussion there is no preference among these. Note that with Fig. 1 two conductor units which lie adjacent in the first piece of cable connect to conductor units separated by a count of one (adjacent) in the second piece. With Fig. 2 conductor units which lie adjacent in the first piece connect to conductor units separated by a count of two in the second piece. With Fig. 3 conductor units which lie adjacent in the first piece connect to conductor units separated by a count of three in the second piece. Splices made in accordance with the schemes of Figs. 1, 2, or 3

will be described as made with a "spread of one," a "spread of two," of a "spread of three," respectively. It is readily shown that for a spread number s to be applicable to cable of N units it is necessary and sufficient that s be prime relative to N.

Figure 4 shows the splicing of six pieces of eleven-unit cable through

$$
\begin{array}{rrrrrr}
1 \rightarrow & 1 \rightarrow & 1 \rightarrow & 1 \rightarrow & 1 \rightarrow & 1 \\
2 \rightarrow & 3 \rightarrow & 5 \rightarrow & 9 \rightarrow & 6 \rightarrow & 11 \\
3 \rightarrow & 5 \rightarrow & 9 \rightarrow & 6 \rightarrow & 11 \rightarrow & 10 \\
4 \rightarrow & 7 \rightarrow & 2 \rightarrow & 3 \rightarrow & 5 \rightarrow & 9 \\
5 \rightarrow & 9 \rightarrow & 6 \rightarrow & 11 \rightarrow & 10 \rightarrow & 8 \\
6 \rightarrow & 11 \rightarrow & 10 \rightarrow & 8 \rightarrow & 4 \rightarrow & 7 \\
7 \rightarrow & 2 \rightarrow & 3 \rightarrow & 5 \rightarrow & 9 \rightarrow & 6 \\
8 \rightarrow & 4 \rightarrow & 7 \rightarrow & 2 \rightarrow & 3 \rightarrow & 5 \\
9 \rightarrow & 6 \rightarrow & 11 \rightarrow & 10 \rightarrow & 8 \rightarrow & 4 \\
10 \rightarrow & 8 \rightarrow & 4 \rightarrow & 7 \rightarrow & 2 \rightarrow & 3 \\
11 \rightarrow & 10 \rightarrow & 8 \rightarrow & 4 \rightarrow & 7 \rightarrow & 2
\end{array}
$$

Fig. 4

the successive application of five consecutive identical splices, each with a spread of two. Following the "key" of the first and second columns, the succeeding columns are written down immediately. Scrutiny of the sequences of numbers appearing in the several columns reveals at once the fundamental properties of the spread. For a cable of N units these are:

1. Successive applications of a spread of s for n times result in a spread of s^n.

2. A spread of minus s is equivalent effectively to a spread of plus s.

3. A spread of $KN + s$ (K is an integer: positive, negative, or zero) is the same effectively as a spread of s.

The problem of achieving the minimum possible recurrence of same-layer adjacencies among conductor units through the application of successive similar splices in accordance with a simple spread now may be stated formally in the terminology and symbols of number theory. If N, an integer, is the number of conductor units in the cable, and if s, an integer prime to N, is the spread number used, then it is required to find a value for s for which the companion relations

$$s^d \equiv \pm 1 \pmod{N},$$

$$s^b \not\equiv \pm 1 \pmod{N}, \quad b < d$$

determine the largest possible integer d.

From the foregoing introductory discussion it should be noted that

values for N less than 5 are of no significance to this problem. In the analysis which follows, therefore, no particular effort has been made to render the general conclusions capable of extension to these extreme and trivial cases.

It is necessary at this point to recall and introduce certain working material. First, there is the established theorem that every positive integer N greater than unity can be represented in one and only one way in the form

$$N = p_1^{\alpha_1} p_2^{\alpha_2} \cdots p_t^{\alpha_t},$$

where $p_1, p_2, \cdots, p_t$ are different primes and $\alpha_1, \alpha_2, \cdots, \alpha_t$ are positive integers. Then there is the familiar number theory function $\phi(N)$ which indicates the number of positive integers not greater than N and prime to N.* If p is a prime number and α is a positive integer, then

$$\phi(p^\alpha) = p^{\alpha-1}(p - 1);$$

also

$$\phi(p_1^{\alpha_1} p_2^{\alpha_2} \cdots p_t^{\alpha_t}) = \phi(p_1^{\alpha_1}) \cdot \phi(p_2^{\alpha_2}) \cdot \cdots \cdot \phi(p_t^{\alpha_t}),$$

where $p_1, p_2, \cdots, p_t$ are different primes.

Then there is the λ-function defined in terms of the ϕ-function as follows:

$$\lambda(2^\alpha) = \phi(2^\alpha) \text{ for } \alpha = 0, 1, 2,$$

$$\lambda(2^\alpha) = \frac{\phi(2^\alpha)}{2} \text{ for } \alpha > 2,$$

$$\lambda(p^\alpha) = \phi(p^\alpha) \text{ for } p \text{ an odd prime},$$

$$\lambda(2^{\alpha_1} p_2^{\alpha_2} p_3^{\alpha_3} \cdots p_t^{\alpha_t}) = M,$$

where M is the least common multiple of

$$\lambda(2^{\alpha_1}),\ \lambda(p_2^{\alpha_2}),\ \lambda(p_3^{\alpha_3}),\ \cdots,\ \lambda(p_t^{\alpha_t}),$$

$2, p_2, p_3, \cdots, p_t$ being different primes.† Finally, it is established that for two relatively prime integers s and N the value $\lambda(N)$ is the largest possible for the exponent m for which the relations

$$s^m \equiv 1 \pmod{N},$$

$$s^n \not\equiv 1 \pmod{N}, \quad n < m,$$

* Euler, "Novi Comm. Ac. Petrop.," 1760–61, p. 74. Carmichael, "The Theory of Numbers," John Wiley & Sons, Inc., 1914, pp. 30–32. Dickson, "Introduction to the Theory of Numbers," Univ. of Chicago Press, 1929, Chap. 1.

† Cauchy, *Comptes Rendus*, Paris, 1841, pp. 824–845. Carmichael, p. 53.

will hold, and that a value for s belonging to this exponent does exist.*

Here it is convenient to consider separately numbers of the two classes—those for which $\lambda(N) = \phi(N)$ and those for which $\lambda(N) < \phi(N)$. For numbers of the first class established theorems may be drawn upon to furnish a complete analysis. For numbers of the second class, however, it will be necessary to extend a bit beyond the ground covered by previous workers, and the steps will be given in considerable detail. This procedure coupled with the inherent complexity will render the treatment for the latter class much less compact and elegant than that for the former.

Case I. $\lambda(N) = \phi(N)$.

From the defined relation between the ϕ-function and the λ-function it follows that numbers of the class such that $\lambda(N) = \phi(N)$ are confined to the values

$$1,\ 2,\ 4,\ p^\alpha, \quad \text{and} \quad 2p^\alpha,$$

where p is an odd prime and α is a positive integer.† For a number N of this class it is established that there exists a set of $\phi(\phi(N))$ numbers r, such that

$$(1) \qquad r^{\lambda(N)} \equiv 1 \pmod{N} \quad \text{and}$$

$$(2) \qquad r^n \not\equiv 1 \pmod{N}, \quad n < \lambda(N), \quad \lambda(N) = \phi(N).$$

Such a number is known as a "primitive root" of N.‡ From the properties of the primitive root r of the number N as defined by relations (1), (2) it follows readily that

$$(3) \qquad r^{\lambda(N)/2} \equiv -1 \pmod{N},$$

$$(4) \qquad r^n \not\equiv \pm 1 \pmod{N}, \quad 0 < n < \frac{\lambda(N)}{2}, \quad \frac{\lambda(N)}{2} < n < \lambda(N).$$

First there will be considered the companion relations

$$(5) \qquad s^d \equiv -1 \pmod{N},$$

$$(6) \qquad s^b \not\equiv \pm 1 \pmod{N}, \quad b < d,$$

and, from comparison with relations (3) and (4), these clearly are satisfied for s a primitive root of N and for $d = \lambda(N)/2$. That no

* Carmichael, p. 54 and pp. 61–63.

† Carmichael, p. 71.

‡ Gauss, "Disquisitiones Arithmeticae," Art. 52–55. Carmichael, pp. 65–71. Dickson, Sec. 17.

value for d greater than $\lambda(N)/2$ is possible is evident immediately. For let a be any integer prime to N. Then for some exponent k

$$r^k \equiv a \pmod N \quad \text{and}$$

$$a^{\lambda(N)/2} \equiv r^{k\lambda(N)/2} \equiv \pm 1 \pmod N.$$

Next there will be considered the companion relations

$$s^d \equiv 1 \pmod N, \tag{7}$$

$$s^b \not\equiv \pm 1 \pmod N, \quad b < d. \tag{8}$$

The reasoning just above shows that d cannot be greater than $\lambda(N)/2$. Suppose for the moment that d has this greatest possible value $\lambda(N)/2$. Relations (7), (8) then become

$$s^{\lambda(N)/2} \equiv 1 \pmod N,$$

$$s^b \not\equiv \pm 1 \pmod N, \quad b = 1, 2, 3, \cdots, \lambda(N)/2 - 1.$$

These relations may be written

$$(s^{1/2})^{\lambda(N)} \equiv 1 \pmod N, \tag{9}$$

$$(s^{1/2})^{2b} \not\equiv \pm 1 \pmod N, \quad 2b = 2, 4, 6, \cdots, \lambda(N) - 2. \tag{10}$$

Now relations (9), (10), will be compatible with relations (1), (2), (3), (4) only if $\lambda(N)/2$ is an odd number, for otherwise the restrictions of relation (10) applying to the even numbered exponents from 2 to $\lambda(N) - 2$ inclusive would be in conflict with relation (3). For $\lambda(N)/2$ an odd number, then, relations (9), (10), are satisfield for $s^{1/2}$ a primitive root of N. Consequently, with relations (7), (8), $\lambda(N)/2$ is the largest possible value for the exponent d, and a value for s equal to the square of a primitive root of N permits this to be attained.

Case II. $\lambda(N) < \phi(N)$.

The inquiry for this case will be divided into four parts. In general $N = p_1^{\alpha_1}p_2^{\alpha_2}p_3^{\alpha_3} \cdots p_t^{\alpha_t}$ where $p_1, p_2, p_3, \cdots, p_t$ are different primes.

(a) First will be considered the case where $p_1, p_2, p_3, \cdots, p_t$ are all odd primes. Then $\lambda(N)$ is the least common multiple of $\lambda(p_1^{\alpha_1})$, $\lambda(p_2^{\alpha_2})$, $\lambda(p_3^{\alpha_3})$, $\cdots$, $\lambda(p_t^{\alpha_t})$. Suppose now that the highest power of 2 dividing any of the λ's divides $\lambda(p_i^{\alpha_i})$. If this same power of 2 divides more than one of the λ's, arbitrarily select $\lambda(p_i^{\alpha_i})$ as one of them. Then this power of 2 will be exactly that occurring in $\lambda(N)$. Now arbitrarily select p_j as any one of the odd primes other than p_i. Then clearly

$\lambda(N)$ will also be the least common multiple of

$$\lambda(p_1^{\alpha_1}),\ \lambda(p_2^{\alpha_2}),\ \cdots,\ \lambda(p_{j-1}^{\alpha_{j-1}}),\ \frac{\lambda(p_j^{\alpha_j})}{2},\ \lambda(p_{j+1}^{\alpha_{j+1}}),\ \cdots,\ \lambda(p_t^{\alpha_t}).$$

Now take

$$\begin{aligned} r &\equiv g_j^2 \pmod{p_j^{\alpha_j}}, \quad g_j \text{ a primitive root of } p_j^{\alpha_j}, \\ &\equiv g_k \pmod{p_k^{\alpha_k}}, \quad g_k \text{ a primitive root of } p_k^{\alpha_k}, \\ &\qquad k = 1, 2, 3, \cdots, j-1, j+1, \cdots, t. \end{aligned}$$

The r thus chosen must be prime to each of the prime factors of N, and hence must be prime to N. Consequently it is known that

$$r^{\lambda(N)} \equiv 1 \pmod{N}.$$

Suppose that m is the smallest exponent for which the congruence

$$r^m \equiv 1 \pmod{N}$$

is true. Then it is noted that the chosen r is such that m must be a multiple of

$$\lambda(p_1^{\alpha_1}),\ \lambda(p_2^{\alpha_2}),\ \cdots,\ \lambda(p_{j-1}^{\alpha_{j-1}}),\ \frac{\lambda(p_{1j}^{\alpha_j})}{2},\ \lambda(p_{j+1}^{\alpha_{j+1}}),\ \cdots,\ \lambda(p_t^{\alpha_t}),$$

and the least multiple common to these is, of course, $\lambda(N)$. Therefore it can be written that

$$\begin{aligned} r^{\lambda(N)} &\equiv 1 \pmod{N}, \\ r^b &\not\equiv 1 \pmod{N}, \quad b < \lambda(N). \end{aligned}$$

Now suppose that for some exponent n less than $\lambda(N)$

$$r^n \equiv -1 \pmod{N}.$$

Then

$$r^{2n} \equiv 1 \pmod{N},$$

and if n is less than $\lambda(N)$, $2n$ is less than $2\lambda(N)$ and can only be equal to $\lambda(N)$. It would necessarily follow then that

$$r^{\lambda(N)/2} \equiv -1 \pmod{N},$$

and it would follow in turn that

$$r^{\lambda(N)/2} \equiv -1 \pmod{p_j^{\alpha_j}}.$$

However, r has been chosen such that

$$r^{\lambda(N)/2} \equiv (g_j^2)^{\lambda(N)/2} \equiv g_j^{\lambda(N)} \equiv 1 \pmod{p_j^{\alpha_j}}.$$

This last relation is incompatible with the one immediately above, and it must be concluded that the assumption

$$r^n \equiv -1 \pmod{N}, \quad n < \lambda(N)$$

is false, and that for the r that has been chosen

$$r^{\lambda(N)} \equiv 1 \pmod{N},$$
$$r^b \not\equiv \pm 1 \pmod{N}, \quad b < \lambda(N)$$

and no exponent greater than $\lambda(N)$ is possible.

(b) Next will be considered the case where $p_1 = 2$, $\alpha_1 = 1$, and $p_2, p_3, \cdots, p_t$ are all odd primes. Select p_i as above and take p_j different from 2. Then take

$$\begin{aligned} r &\equiv 1 \pmod{2}, \\ &\equiv g_j^2 \pmod{p_j^{\alpha_j}}, \quad g_k \text{ a primitive root of } p_j^{\alpha_j}, \\ &\equiv g_k \pmod{p_k^{\alpha_k}}, \quad g_k \text{ a primitive root of } p_k^{\alpha_k}, \\ &\qquad k = 2, 3, 4, \cdots, j-1, j+1, \cdots, t, \end{aligned}$$

and the same line of reasoning may be repeated and the same conclusions reached as under part (a) above.

(c) Next will be considered the case where $p_1 = 2$, $\alpha_1 = 2$ and $p_2, p_3, \cdots, p_t$ are all odd primes. Since $\lambda(2^2) = 2$ take p_i different from 2, and for simplicity take p_j as 2. Then take

$$\begin{aligned} r &\equiv 1 \pmod{4}, \\ &\equiv g_k \pmod{p_k^{\alpha_k}}, \quad g_k \text{ a primitive root of } p_k^{\alpha_k}, \\ &\qquad k = 2, 3, 4, \cdots, t \end{aligned}$$

and the same line of reasoning may be repeated and the same conclusions reached as under part (a) above.

(d) Finally will be considered the case where $p_1 = 2$, $\alpha_1 > 2$, and $p_2, p_3, \cdots, p_t$ are all odd primes. Now 5 has the property that

$$5^{\lambda(2^{\alpha_1})} \equiv 1 \pmod{2^{\alpha_1}},$$
$$5^b \not\equiv \pm 1 \pmod{2^{\alpha_1}}, \quad \alpha_1 > 2, \quad b < \lambda(2^{\alpha_1}).$$

So by taking

$$r \equiv 5 \pmod{2^{\alpha_1}},$$
$$\equiv g_k \pmod{p_k^{\alpha_k}}, \quad g_k \text{ a primitive root of } p_k^{\alpha_k},$$
$$k = 2, 3, 4, \cdots, t$$

it is concluded immediately that

$$r^{\lambda(N)} \equiv 1 \pmod{N},$$
$$r^b \not\equiv \pm 1 \pmod{N}, \quad b < \lambda(N).$$

The preceding formal analysis for Case I and Case II may be summed up as having established the following general theorem:

If N is a given positive integer and if s is an integer prime to N, then the largest possible exponent d for which the companion congruencial relations

$$s^d \equiv \pm 1 \pmod{N},$$
$$s^b \not\equiv \pm 1 \pmod{N}, \quad b < d$$

will be true is $\lambda(N)/2$ for numbers such that $\lambda(N) = \phi(N)$ and is $\lambda(N)$ for numbers such that $\lambda(N) < \phi(N)$, and a value for s belonging to this exponent in each instance does exist.

In order to apply the foregoing results to a practical case Table I has been prepared. In the left-hand column appear the numbers 5 to 139, inclusive. In the next column is listed for each number the value of $\lambda(N)/2$ or of $\lambda(N)$, depending upon whether $\lambda(N) = \phi(N)$ or $\lambda(N) < \phi(N)$. In the final column there is listed for each number a suitable value for the spread. There appears to be no advantage of one spread figure over another, and the listing of additional acceptable values is omitted in the interest of economy of space. For the numbers for which $\lambda(N) = \phi(N)$ and for which $\lambda(N)/2$ is odd care has been taken that the listed spread figures are primitive roots, and not the squares of primitive roots which were shown to be equally acceptable. This fact will be recalled later.

It was shown earlier that $[(N-1)/2]$ successive cable lengths would be the maximum possible number for an extended conductor unit to traverse without incurring repetition of at least one of the same-layer adjacencies which occurred in the first of these lengths. On referring to Table I it is seen that only for the prime numbers is this maximum attainable. The prime numbers are distinguished by the fact that for them $\lambda(N)/2 = (N-1)/2$, and each has been indicated by an asterisk. The composite numbers are seen to yield quite inferior results in general.

TABLE I

For each N there is listed the value d and a value s for which the companion relations
$s^d \equiv \pm 1 \pmod{N} \qquad s^b \not\equiv \pm 1 \pmod{N},\ b < d$
determine the largest possible integer d.

N	d	s	N	d	s	N	d	s
5*	2	2	50	10	3	95	36	2
6	1	5	51	16	5	96	8	5
7*	3	3	52	12	7	97*	48	5
8	2	3	53*	26	2	98	21	3
9	3	2	54	9	5	99	30	5
10	2	3	55	20	2	100	20	3
11*	5	2	56	6	3	101*	50	2
12	2	5	57	18	5	102	16	5
13*	6	2	58	14	3	103*	51	5
14	3	3	59*	29	2	104	12	7
15	4	2	60	4	7	105	12	2
16	4	3	61*	30	2	106	26	3
17*	8	3	62	15	3	107*	53	2
18	3	5	63	6	2	108	18	5
19*	9	2	64	16	3	109*	54	6
20	4	3	65	12	3	110	20	3
21	6	2	66	10	5	111	36	2
22	5	7	67*	33	2	112	12	3
23*	11	5	68	16	3	113*	56	3
24	2	5	69	22	2	114	18	5
25	10	2	70	12	3	115	44	2
26	6	7	71*	35	7	116	28	3
27	9	2	72	6	5	117	12	2
28	6	5	73*	36	11	118	29	11
29*	14	2	74	18	5	119	48	3
30	4	7	75	20	2	120	4	7
31*	15	3	76	18	21	121	55	2
32	8	3	77	30	2	122	30	7
33	10	5	78	12	7	123	40	7
34	8	3	79*	39	3	124	30	7
35	12	2	80	4	3	125	50	2
36	6	5	81	27	2	126	6	11
37*	18	2	82	20	7	127*	63	3
38	9	3	83*	41	2	128	32	3
39	12	2	84	6	5	129	42	14
40	4	3	85	16	3	130	12	3
41*	20	6	86	21	3	131*	65	2
42	6	11	87	28	2	132	10	5
43*	21	3	88	10	3	133	18	2
44	10	3	89*	44	3	134	33	7
45	12	2	90	12	7	135	36	2
46	11	5	91	12	2	136	16	3
47*	23	6	92	22	3	137*	68	3
48	4	5	93	30	13	138	22	7
49	21	5	94	23	5	139*	69	2

* The asterisk indicates a prime number.

For the benefit of the practical worker there must be described a slight deviation from the second simplifying restriction imposed at the beginning which will permit the maximum possibility to be realized if N is one plus a prime number. This artifice is based upon the fact

that for r a primitive root of a number N for which $\lambda(N) = \phi(N)$ and in particular for a prime number N

$$r^{\lambda(N)/2} \equiv -1 \pmod{N}.$$

This means that $\lambda(N)/2$ consecutive splices with a spread r result in a spread of minus one. It is readily shown that this in turn means that there will be two conductor units No. b and No. $b + 1$ in the first length of cable which ultimately will be extended to connect respectively to units No. $b + 1$ and No. b. In Fig. 4 two units No. 6 and No. 7 meet this requirement. To illustrate the use of this artifice it will be supposed that a cable of 12 units is to be spliced. Referring to Fig. 4. for guidance, the arrangement shown in Fig. 5 is set up readily. The first two columns indicate the splicing assignment, and the succeeding columns are then derived from these. The eleven units 1, 2, $\cdots$, 5, 6, 8, 9, $\cdots$, 11, 12 are assigned exactly in conformity with the scheme of Fig. 4, ignoring the break in sequence between No. 6 and No. 8. Unit No. 7 is then simply spliced to itself throughout.

$$\begin{array}{rcrcrcrcrcr}
1 & \rightarrow & 1 & \rightarrow & 1 & \rightarrow & 1 & \rightarrow & 1 & \rightarrow & 1 \\
2 & \rightarrow & 3 & \rightarrow & 5 & \rightarrow & 10 & \rightarrow & 6 & \rightarrow & 12 \\
3 & \rightarrow & 5 & \rightarrow & 10 & \rightarrow & 6 & \rightarrow & 12 & \rightarrow & 11 \\
4 & \rightarrow & 8 & \rightarrow & 2 & \rightarrow & 3 & \rightarrow & 5 & \rightarrow & 10 \\
5 & \rightarrow & 10 & \rightarrow & 6 & \rightarrow & 12 & \rightarrow & 11 & \rightarrow & 9 \\
6 & \rightarrow & 12 & \rightarrow & 11 & \rightarrow & 9 & \rightarrow & 4 & \rightarrow & 8 \\
7 & \rightarrow & 7 & \rightarrow & 7 & \rightarrow & 7 & \rightarrow & 7 & \rightarrow & 7 \\
8 & \rightarrow & 2 & \rightarrow & 3 & \rightarrow & 5 & \rightarrow & 10 & \rightarrow & 6 \\
9 & \rightarrow & 4 & \rightarrow & 8 & \rightarrow & 2 & \rightarrow & 3 & \rightarrow & 5 \\
10 & \rightarrow & 6 & \rightarrow & 12 & \rightarrow & 11 & \rightarrow & 9 & \rightarrow & 4 \\
11 & \rightarrow & 9 & \rightarrow & 4 & \rightarrow & 8 & \rightarrow & 2 & \rightarrow & 3 \\
12 & \rightarrow & 11 & \rightarrow & 9 & \rightarrow & 4 & \rightarrow & 8 & \rightarrow & 2
\end{array}$$

Fig. 5

Undoubtedly there are other equally acceptable artifices for extending further the practical scope of the simple results. The prime numbers and the prime numbers plus one constitute nearly fifty percent of all numbers in the range in which the practical worker is likely to be interested, however, and when it is borne in mind that normally he has latitude in his choice of N it is seen that the material here presented is adequate for his needs.

The writer is indebted to Dr. D. H. Lehmer for pertinent suggestions. The entire treatment for the case of numbers for which $\lambda(N) < \phi(N)$ follows a line of attack suggested by Mr. Marshall Hall, and but for his helpful interest this presentation would have been lacking in formal completeness.